Effects of Overtraining, Problem Shifts, and Probabilistic Reinforcement in Discrimination Learning: Predictions of an Attentional Model

M. Frank Norman
UNIVERSITY OF PENNSYLVANIA

1. INTRODUCTION

This chapter is concerned with a model for discrimination learning that incorporates interacting perceptual- and response-learning processes. The subject learns to pay attention to the relevant stimulus dimension. Simultaneously, he learns to bias his responses toward the correct value or cue along that dimension.

The concept of selective attention to a stimulus dimension has only received full respectability within experimental psychology over the last two decades. The phenomenon that implicates an attentional mechanism most directly is this: intradimensional shifts are generally learned more easily than extradimensional shifts (Estes, 1970, pp. 169–170). In the first case, the subject is shifted to a new problem with the same relevant and irrelevant dimensions (e.g., shape and color of geometrical forms), while the relevant and irrelevant dimensions are interchanged in extradimensional shifts. Specific response transfer can be eliminated in both paradigms by introducing new cues (e.g., square and cross in place of circle and triangle) on both dimensions. Thus the advantage of the intradimensional shift must be due to transfer of attention to the previously relevant dimension into the second phase of the experiment.

From Contemporary Developments In Mathematical Psycholgy, Volume I, edited by D. H. Krantz et al. W. H. Freeman and Company.

The prediction of the relative difficulty of these two shifts is so straightforward that it may be regarded as following from 'attention theory' rather than from any particular 'two-process model.' However, one need not look far in the literature on discrimination learning to find important phenomena of sufficient subtlety that careful analysis is greatly facilitated by a precise model. Two of these are the relative difficulty of extradimensional and reversal shifts, and the effect of overtraining on reversal. In a reversal shift, the relevant and irrelevant dimensions remain the same, as do the cues on these dimensions, but the correct and incorrect cues (on the relevant dimension) are interchanged.

Overtraining sometimes facilitates reversal (the overlearning reversal effect or ORE) and sometimes retards it (Mackintosh, 1969). Children learn reversal more quickly than extradimensional shift (Campione, 1971), but Kelleher's (1956) rats learned extradimensional shift more easily. As we see in Sections 8 and 9, any of these results can be explained by suitable choice of the parameter values in the model considered. For example, the relative rates of perceptual and response learning must be chosen appropriately. These choices make good sense psychologically, and yield testable predictions for other experiments.

Reward is perfectly correlated with one cue on the relevant dimension in all of the preceding experimental paradigms. Other experiments have used probabilistic reinforcement, with attendant imperfect correlation. For example, in a brightness discrimination with position irrelevant, black might be correct with probability π, $\frac{1}{2} < \pi < 1$, while white is correct on the remaining trials. On such schedules, subjects' asymptotic proportions of black choices almost always fall between π and 1, and values in the upper part of this range are common in experiments with animals, even when response correction is allowed after errors. The upper limit 1, which represents optimal behavior, is approached under noncorrection procedures (Sutherland & Mackintosh, 1971, pp. 405–409).

In summary, performance in these experiments ranges from fairly efficient to very efficient. We see in Section 10 that these levels of efficiency are consistent with the model, provided that correct responses produce more perceptual and response learning than incorrect ones. This imposes a heavy constraint on the model's parameters. However, this constraint is consistent with those imposed by shift and overtraining studies.

The scope and power of attentional concepts are generally appreciated in psychology today, but our understanding of these ideas lacks depth and precision. It is very difficult to form a clear idea of the possibilities inherent in two-process conceptions of discrimination learning without specifying a mathematical model and exploring the effects of variations in its parameters. These are my objectives in this chapter.

2. EXPERIMENTS

It is often easiest to understand an abstract psychological model within the context of a specific experiment. I describe two typical experiments in this section: learning of a brightness discrimination by a rat in a *jumping stand*, and learning of a color discrimination by a child in a modified *Wisconsin General Test Apparatus* (WGTA). Discussion of the model in later sections is cast primarily in the terminology of a jumping stand, and secondarily in that of WGTA. The reader should have no difficulty transposing the discussion to other settings.

(1) *Jumping stand.* (See Sutherland & Mackintosh, 1971, Fig. 2.1, p. 25). On each trial, the rat must jump from a central platform to a right- or left-hand ledge. Above each ledge is a window in which a stimulus card is displayed. One card is black, the other is white, and the black card appears on the left on a random 50 percent of trials. Behind the window is a platform with a tray of food. One of the cards, say the black one, can be pushed aside permitting access to food. The window containing the white card is always locked.

(2) *WGTA.* (See Zeaman & House, 1963, Fig. 5-1, p. 160.) The child moves one of two stimulus objects on a tray before him in order to expose a shallow well. On half the trials, the objects are a red triangle and a green circle; on the remaining trials, they are a red circle and a green triangle. Each object appears equally often on the child's left and right. There is a piece of candy under the red object, but nothing under the green one.

In either of these experiments, response *correction* may be permitted. Thus a child who makes an error can move the other object and obtain candy. In a noncorrection procedure, no such second choice is allowed.

Probabilistic reinforcement. In these examples, reward is perfectly correlated with one cue on the relevant dimension. This is *consistent reinforcement.* In probabilistic reinforcement experiments, this correlation is imperfect. I have already mentioned the possibility that reward might be associated with one cue on a proportion π of the trials and with the other cue on the same dimension on the remaining trials. This is called *noncontingent reinforcement,* in recognition of the fact that the correct cue on any trial is determined by the experimenter, without regard to the subject's response.

Consider an experiment of this type in a jumping stand, and let π_B and π_W be the probabilities that black (B) and white (W) responses are correct:

$$\pi_B = P(\text{B correct}), \quad \pi_W = P(\text{W correct}).$$

Then

$$\pi_B = 1 - \pi_W = \pi. \tag{1}$$

In noncorrection experiments, there is no need to unlock exactly one window on each trial. Thus π_B and π_W can be varied independently. These parameters determine a *probabilistic reinforcement schedule*. In addition to noncontingent schedules, which satisfy Equation 1, we will be interested in *symmetric schedules*, defined by

$$\pi_B = \pi_W = \zeta. \tag{2}$$

Under such a schedule, brightness and position are both irrelevant (i.e., useless) dimensions. *Extinction* is the special case where no response is ever rewarded ($\zeta = 0$). The only probabilistic reinforcement schedule that we exclude at the onset is the one on which all responses are correct ($\zeta = 1$). In other words, *it is always assumed that*

$$\pi_B < 1 \quad \textit{or} \quad \pi_W < 1. \tag{3}$$

For any number p, let $p' = 1 - p$. The quantity

$$\ell = \pi'_W/(\pi'_W + \pi'_B) \tag{4}$$

is associated with probability matching, in the sense that $P(B) = \ell$ is equivalent to

$$P(B) = P(\text{reinforcement of B}). \tag{5}$$

"Reinforcement of B" means that the rat jumps to an unlocked black window or a locked white window, hence

$$P(\text{reinf. of B}) = P(B)\pi_B + (1 - P(B))\pi'_W.$$

It follows that

$$\begin{aligned} P(\text{reinf. of B}) - P(B) &= -P(B)\pi'_B + (1 - P(B))\pi'_W \\ &= \pi'_W - P(B)(\pi'_W + \pi'_B) \\ &= (\pi'_W + \pi'_B)(\ell - P(B)). \end{aligned}$$

By Equation 3, $\pi'_W + \pi'_B > 0$, so $P(B) = \ell$ is equivalent to Equation 5, as claimed. For noncontingent schedules this equivalence is trivial, since

$$P(\text{reinf. of B}) = \pi = \ell.$$

Note that consistent reinforcement of black is the special case of probabilistic reinforcement defined by $\pi_B = 1$ and $\pi_W = 0$.

3. THE MODEL

The model treated in this chapter was proposed by Zeaman and House (1963) and Lovejoy (1966). I call it the *Zeaman-House-Lovejoy* or *ZHL model.* Some special features of the formulations in these two papers will be noted after the model has been described in the context of a jumping-stand experiment with probabilistic reinforcement.

On any trial, the rat attends to brightness (br) or to position (po), but not to both. His probability of attending to brightness is denoted v:

$$v = P(\text{br}).$$

If he attends to brightness, he chooses black (B) rather than white (W) with conditional probability y:

$$y = P(\text{B} \mid \text{br}).$$

If he attends to position, he chooses B with probability 0.5. The variables v and y or, equivalently, the composite variable

$$x = (v, y),$$

determines the subject's 'state of learning' on any trial. Given x, the probability of B is

$$P(\text{B} \mid x) = yv + 2^{-1}v', \tag{6}$$

where

$$v' = 1 - v.$$

I now describe how v and y change on each trial. This depends on what the subject attends to, what response he makes, and whether this response is correct (C) or incorrect (I). The variable y changes only when the subject attends to brightness. It increases after a correct B or an incorrect W and decreases otherwise. The quantity v increases when the subject attends to brightness and is correct, or attends to position and is incorrect. On other trials it decreases.

These changes are effected by linear transformations. Thus if the subject attends to brightness, jumps to B, and is rewarded,

$$\Delta y = \theta_1 y'$$

for some $0 < \theta_1 < 1$. The increment in y is a proportion θ_1 of the maximum possible increment $y' = 1 - y$. Similarly,

$$\Delta y = -\theta_2 y$$

TABLE 1
Events, associated transformations, and probabilities for the ZHL model

Event	Δv	Δy	Probability
brBC	$\varphi_1 v'$	$\theta_1 y'$	$vy\pi_B$
brBI	$-\varphi_2 v$	$-\theta_2 y$	$vy\pi'_B$
brWC	$\varphi_1 v'$	$-\theta_1 y$	$vy'\pi_W$
brWI	$-\varphi_2 v$	$\theta_2 y'$	$vy'\pi'_W$
poBC	$-\varphi_3 v$	0	$v'\pi_B/2$
poBI	$\varphi_4 v'$	0	$v'\pi'_B/2$
poWC	$-\varphi_3 v$	0	$v'\pi_W/2$
poWI	$\varphi_4 v'$	0	$v'\pi'_W/2$

if he attends to brightness, jumps to B, and is not rewarded. Table 1 lists the values of Δv and Δy corresponding to the eight possible combinations of attention, choice, and outcome. Any such combination is called an 'event.' Probabilities of these events are also listed.

The fact that there are four φ_is in Table 1 instead of eight, and two θ_js instead of four, reflects an assumption of black-white symmetry. Thus brBC and brWC have identical effects on v. Also, brBC has the same effect on $y = P(\text{B})$ $[\Delta y = \theta_1(1 - y)]$ that brWC has on $y' = P(\text{W})$ $[\Delta y' = \theta_1(1 - y')]$.

It is assumed throughout the chapter that

$$0 < \varphi_i < 1 \quad \textit{and} \quad 0 < \theta_j < 1 \tag{7}$$

for all i and j. The occurrence of near-optimal performance under probabilistic reinforcement points to very small values of the nonreward parameters φ_2, φ_4, and θ_2 for noncorrection procedures. The exclusion of zero values of these parameters represents the most notable loss of generality in Equation 7.

I turn now to the relationship of the present formulation to its predecessors. Zeaman and House (1963) were interested in discrimination learning by retarded children. The model they described could handle an arbitrary number of stimulus dimensions. I restrict attention to the case of only two dimensions, because it appears to be much more tractable mathematically than the general case. An important feature of the Zeaman-House formulation is that it specifies a learning process for the probability z of choosing left, given attention to position, and thus describes the laterality as well as the brightness of a subject's choice. In particular, it can predict certain types of 'position habits.' [1] However, since the black card appears on the left and

[1] If a rat is switched from a problem with left correct to one with black correct, the model predicts better performance when black is on the left than when it is on the right. Graf and Tighe (1971), who observed an analogous difference in an experiment with turtles, thought it was at variance with attention theory.

right equally often, $P(\mathrm{B})$ does not depend on z, and the model can be 'reduced' to the form given in Table 1.

Zeaman and House assumed that $\varphi_1 = \varphi_2 = \varphi_3 = \varphi_4$ and $\theta_1 = \theta_2$. A number of implications of this extremely restrictive condition are presented in Section 7.

Lovejoy (1966) was concerned with the ORE in rats. Some of his work is reviewed in Section 9. My description of his model differs significantly from his in only one respect. He avoids restriction to experiments with a single irrelevant dimension by simply not specifying what the subject is doing when he is not attending to the relevant dimension. In effect, multidimensional problems are reduced to two dimensions by lumping together all irrelevant dimensions. This approach to multidimensional problems is not compatible with that of Zeaman and House.

How can the ZHL model be applied to the WGTA experiment with its two irrelevant dimensions, form and position? In the discussion of shifts in Section 8, we simply ignore position, on the assumption that pretraining and instructions have suppressed it within the child's attentional repertoire. An alternative suggested by Lovejoy's formulation is to describe the focus of attention as 'color' or 'something else.'

I now introduce some notations and note some simple equalities that are needed in subsequent sections. Let V_n, Y_n, and X_n be the values of v, y, and $x = (v, y)$ on trial n, where $n = 0, 1, 2, \ldots$. Clearly V_n and Y_n are random variables, and X_n is a random vector. Let

$$v_n = E(V_n) \quad \text{and} \quad y_n = E(Y_n)$$

be the expectations of V_n and Y_n. A subscript n on br, po, B, W, C, or I indicates an occurrence on trial n. For example, B_n means 'the rat jumps to black on trial n.' Clearly $v_n = P(\mathrm{br}_n)$, and, by Equation 6,[2]

$$P(\mathrm{B}_n) = E(V_n Y_n) + 2^{-1}v_n'. \tag{8}$$

Similarly,

$$P(\mathrm{W}_n) = E(V_n Y_n') + 2^{-1}v_n'. \tag{9}$$

4. REGULARITY AND ABSORPTION

It is easy to see that a subject's state of learning, X_n, on trial n summarizes all information about preceding trials that is relevant to predicting future

[2] Here we use the fact that, for an arbitrary event B an an arbitrary random vector X, $P(B) = E(P(B|X))$. Similarly, for an essentially arbitrary random variable Y, $E(Y) = E(E(Y|X))$ (Brunk, 1965, p. 94). These basic relations are used in the sequel without further comment.

behavior. Hence this stochastic process is Markovian. Let $K^{(n)}(x, \cdot)$ be the distribution of X_n when $X_0 = x$:

$$K^{(n)}(x, A) = P(X_n \in A \mid X_0 = x).$$

Under the assumptions of Equations 3 and 7, the Markov process X_n is *regular*. This means that, as $n \longrightarrow \infty$, $K^{(n)}(x, \cdot)$ converges (weakly) to a distribution $K^{\infty}(\cdot)$ that does not depend on x. In particular, the limits

$$\begin{aligned} v_\infty &= \lim v_n, \\ y_\infty &= \lim y_n, \\ E(V_\infty Y_\infty) &= \lim E(V_n Y_n), \\ P(\mathrm{B}_\infty) &= \lim P(\mathrm{B}_n) \end{aligned}$$

exist and do not vary with x. In each case, convergence occurs at a geometric rate.

To proceed further it is necessary to note whether the process X_n has any absorbing states. It is easy to show that (1, 1) is absorbing if and only if $\pi_{\mathrm{B}} = 1$, that (1, 0) is absorbing if and only if $\pi_{\mathrm{W}} = 1$, and that no other states are absorbing. Equation 3 excludes the case $\pi_{\mathrm{W}} = \pi_{\mathrm{B}} = 1$ of two absorbing states, which is incompatible with regularity.[3]

(a) Suppose that $\pi_{\mathrm{B}} < 1$ and $\pi_{\mathrm{W}} < 1$, so that there are no absorbing states. In this case, $0 < P(\mathrm{B}_\infty) < 1$. Let $\overline{\mathrm{B}}_{I,J}$ be a single subject's proportion of Bs in the J-trial block beginning with an arbitrary trial I. Then

$$\overline{\mathrm{B}}_{I,J} \longrightarrow P(\mathrm{B}_\infty) \tag{10}$$

with probability one or almost surely (a.s.) as $J \longrightarrow \infty$. In particular, *a subject's asymptotic proportion of jumps to* B *does not depend on his initial perceptual or response biases, or on the idiosyncrasies of his training.* These idiosyncrasies, such as the particular sequence of rewards and nonrewards experienced by the subject, reflect the inherent randomness of the model.

The quantity

$$\rho_j = \lim_{n \to \infty} [P(\mathrm{B}_n \text{ and } \mathrm{B}_{n+j}) - P(\mathrm{B}_n)P(\mathrm{B}_{n+j})]$$

is a measure of the asymptotic dependence between responses j trials apart. Let

$$\sigma^2 = \rho_0 + 2 \sum_{j=1}^{\infty} \rho_j.$$

The law of large numbers in Equation 10 is complemented by the following central limit theorem: $\sigma^2 > 0$, and $\overline{\mathrm{B}}_{I,J}$ is asymptotically normally distributed

[3] If $(1, i)$ is absorbing, and $X_0 = (1, i)$, then $y_\infty = i$. Hence, if both (1, 1) and (1, 0) are absorbing, y_∞ depends on X_0, and X_n is not regular.

with mean $P(B_\infty)$ and variance σ^2/J as $J \longrightarrow \infty$. The quantity σ^2 can be estimated from a single subject's data by standard methods, and this estimate can be used in conjunction with the central limit theorem to make inferences about $P(B_\infty)$ (see Norman, 1971). For example, we can test the hypothesis that $P(B_\infty)$ equals the probability matching asymptote ℓ or some other specified value.

(b) Suppose that $\pi_B = 1$ and $\pi_W < 1$. Then (1, 1) is the only absorbing state, and X_n is an *absorbing process*. This means that $V_n \longrightarrow 1$ and $Y_n \longrightarrow 1$ a.s. Clearly $P(W_\infty) = 0$. Furthermore

$$\#W = \text{total number of Ws over all trials}$$

is finite a.s. and $E(\#W) < \infty$, regardless of the distribution of X_0. Similarly $P(po_\infty) = 0$, $\#po < \infty$ a.s., and $E(\#po) < \infty$.

Thus, when B is always correct but W is not, the rat eventually stops making both overt and perceptual errors (Ws and pos). This highly intuitive prediction is embarrassingly difficult to derive rigorously. It is of greater mathematical than empirical interest.

(c) The case $\pi_B < 1$ and $\pi_W = 1$ parallels (b).

Comments.[4] Norman (1972, Ch. 16) treats the ZHL model with consistent reinforcement. By generalizing arguments in that chapter, it can be shown that the ZHL model with probabilistic reinforcement is *distance diminishing*, and that X_n is a *regular compact Markov process*. Such models and processes are studied at great length in Part I of the same volume. Most of the results in this section follow easily from this general theory. In particular, the bulk of the material under (a) follows from theorems in Chapter 5 via the Corollary to Theorem 6.1.1 (p. 99). We note that the same approach yields analogous properties of uniprocess linear models with no absorbing states (see Norman, 1972, pp. 180–181).

5. CHANGES IN v_n AND y_n

Table 1 describes the changes in the random variables V_n and Y_n that result from various events on trial n. For later computations, we shall need the following formulas for changes in their expectations v_n and y_n:

$$\Delta y_n = E(V_n Q(Y_n)), \tag{11}$$

$$\Delta v_n = E(H(V_n, Y_n)), \tag{12}$$

[4] Most proofs and comments in lieu of proofs are given at ends of sections. All such material can be skipped without loss of continuity.

where

$$Q(y) = \delta yy' + \theta_2(y'\pi'_W - y\pi'_B),$$
$$\delta = (\theta_1 - \theta_2)(\pi_B - \pi_W), \quad (13)$$

$$H(v, y) = v[\varphi_1 v'(y\pi_B + y'\pi_W) - \varphi_2 v(y\pi'_B + y'\pi'_W)]$$
$$+ v'[-\varphi_3 v 2^{-1}(\pi_B + \pi_W) + \varphi_4 v' 2^{-1}(\pi'_B + \pi'_W)]. \quad (14)$$

Proof. We begin by computing $E(\Delta Y_n \mid X_n = x)$. Weighting the values of Δy in the first four rows of Table 1 by their probabilities of occurrence we obtain

$$E(\Delta Y_n \mid X_n = x) = \theta_1 y'vy\pi_B - \theta_2 yvy\pi'_B - \theta_1 yvy'\pi_W + \theta_2 y'vy'\pi'_W$$
$$= v[\theta_1(\pi_B - \pi_W)yy' + \theta_2(y'^2\pi'_W - y^2\pi'_B)]. \quad (15)$$

Now $y^2 = y - yy'$ and $y'^2 = y' - yy'$, hence

$$y'^2\pi'_W - y^2\pi'_B = (\pi'_B - \pi'_W)yy' + y'\pi'_W - y\pi'_B$$
$$= -(\pi_B - \pi_W)yy' + y'\pi'_W - y\pi'_B.$$

Substitution of the latter expression into Equation 15 yields

$$E(\Delta Y_n \mid X_n = x) = vQ(y).$$

Taking expectations on both sides, we obtain

$$E(\Delta Y_n) = E(V_n Q(Y_n)),$$

and Equation 11 follows on noting that $\Delta y_n = E(\Delta Y_n)$.

Similarly, to establish Equation 12 one verifies that

$$E(\Delta V_n \mid X_n = x) = H(v, y).$$

6. SYMMETRIC SCHEDULES

This section considers schedules with $\pi_B = \pi_W = \zeta < 1$. If we take $Y_0 = 1/2$, then all parameters of the model (learning rate, schedule, and initial bias) incorporate B – W symmetry, so we must have $P(B_n) = P(W_n)$ and $y_n = y'_n$; i.e., $P(B_n) = y_n = 1/2$. This yields

$$P(B_\infty) = y_\infty = 1/2 \quad (16)$$

on letting $n \longrightarrow \infty$. However, as noted in the last section, $P(B_\infty)$ and y_∞ do not depend on the initial values of v and y. Consequently, Equation 16 holds whether or not $Y_0 = 1/2$.

The standard extinction paradigm is the special case $\zeta = 0$. However, according to the model any symmetric schedule leads to 'choice extinction,' in the sense that the subject's proportion of Bs over a long block of trials will approach $\frac{1}{2}$.

Let

$$\#_J\text{B} = \text{number of Bs over trials 0 to } J - 1,$$

and let $\#_J$W be the analogous quantity for W. If the subject starts the current phase of the experiment with a brightness preference, $\#_J\text{B} - \#_J\text{W}$ provides a measure of the resulting excess or deficiency in B choices over J trials.

THEOREM 1. *As* $J \longrightarrow \infty$,

$$E(\#_J\text{B} - \#_J\text{W}) \longrightarrow (y_0 - 2^{-1})/\theta_2\zeta'. \tag{17}$$

It is remarkable that the right-hand side of Equation 17 depends on only one of six learning-rate parameters, and does not depend on v_0.

Our main interest in this theorem is its implication for the effect of overtraining on subsequent extinction. If B is consistently reinforced in the preceding phase of the experiment, the final value of $E(Y)$ is an increasing function of the number of trials administered. This final value of $E(Y)$ is the initial value for extinction, denoted y_0 in Equation 17. Thus y_0 is increased by overtraining. Suppose that the number J of extinction trials is sufficiently large that the two sides of Equation 17 can be treated as equal. Then $E(\#_J\text{B} - \#_J\text{W})$ is increased by overtraining. Thus the model predicts that overtraining with B correct tends to increase the excess of B choices in a large number of extinction trials. In other words, *overtraining increases resistance to extinction*. Data supporting this conclusion, but using number of trials to a criterion of extinction as the dependent variable, have been reported by Mackintosh (1963). It is perhaps surprising that this prediction holds for all values of the model's parameters, even those that yield an ORE.

Proof. Since $\delta = 0$, Equation 13 yields

$$Q(y) = \theta_2\zeta'(y' - y).$$

Thus, by Equation 11,

$$\Delta y_n = \theta_2\zeta' E(V_n(Y_n' - Y_n)).$$

Subtracting Equation 8 from Equation 9 we get

$$P(\text{W}_n) - P(\text{B}_n) = E(V_n(Y_n' - Y_n)),$$

so

$$\Delta y_n = \theta_2\zeta'[P(\text{W}_n) - P(\text{B}_n)].$$

Summation over $0 \leq n \leq J - 1$ yields

$$\begin{aligned} y_J - y_0 &= \theta_2\zeta'\left[\sum_{n=0}^{J-1} P(\mathrm{W}_n) - \sum_{n=0}^{J-1} P(\mathrm{B}_n)\right] \\ &= \theta_2\zeta'[E(\#_J\mathrm{W}) - E(\#_J\mathrm{B})] \\ &= \theta_2\zeta' E(\#_J\mathrm{W} - \#_J\mathrm{B}). \end{aligned}$$

The theorem follows on noting that the left-hand side converges to $2^{-1} - y_0$ as $J \longrightarrow \infty$.

7. THE CASE OF ONLY TWO LEARNING RATES

Following Zeaman and House (1963) it is assumed in this section that there are only two learning-rate parameters: one for attentional learning and one for response learning. In other words,

$$\varphi_i = \varphi \quad \text{and} \quad \theta_j = \theta \tag{18}$$

for all i and j. Under this assumption it is possible to calculate $P(\mathrm{B}_\infty)$. When $\pi_\mathrm{B} = 1$, so that $P(\mathrm{B}_\infty) = 1$, there is a simple formula for $E(\#\mathrm{W})$.

Theorem 2 relates $P(\mathrm{B}_\infty)$ to the probability matching asymptote ℓ (see Eq. 4).

THEOREM 2. $P(\mathrm{B}_\infty) = \ell\left[\dfrac{1 + 2\ell'}{1 + 4\ell\ell'}\right].$

It is easily shown that the bracketed ratio is less than 1, so that $P(\mathrm{B}_\infty) < \ell$, for $\frac{1}{2} < \ell < 1$. *Thus, under the assumption of Equation 18, the ZHL model predicts that* $P(\mathrm{B}_\infty)$ *will undershoot the probability matching asymptote.* Zeaman and House (1963, pp. 206, 210) obtained some examples of undershooting in simulations of the multidimensional version their of model. The prediction of undershooting is at variance with the results of numerous simultaneous visual discrimination experiments on animals. Surprisingly, there do not seem to be any reports of comparable experiments on human subjects. In Section 10 we see that the model predicts more efficient performance for other parameter values.

Theorem 3, which assumes Equation 18, gives the only explicit formula for $E(\#\mathrm{W})$ that is presently known.

THEOREM 3. *If* $\pi_\mathrm{B} = 1$ *and* $\pi_\mathrm{W} < 1$,

$$\pi_\mathrm{W}' E(\#\mathrm{W}) = \varphi^{-1}v_0' + 2\theta^{-1}y_0'. \tag{19}$$

This equation is the basis for our comparison of shifts in Section 8 and our study of overtraining in Section 9.

Proofs. Since $\delta = 0$ in Equation 13,

$$Q(y) = \theta(y'\pi_\mathrm{W}' - y\pi_\mathrm{B}'),$$

and Equation 11 yields

$$\theta^{-1}\Delta y_n = E(V_n(Y'_n\pi'_W - Y_n\pi'_B)). \tag{20}$$

According to Equation 14,

$$\begin{aligned}\varphi^{-1}H(v, y) &= v[v'(y\pi_B + y'\pi_W) - v(y\pi'_B + y'\pi'_W)]\\ &\quad + v'[-v2^{-1}(\pi_B + \pi_W) + v'2^{-1}(\pi'_B + \pi'_W)].\end{aligned}$$

Noting that

$$y\pi_B + y'\pi_W = 1 - (y\pi'_B + y'\pi'_W)$$

and

$$2^{-1}(\pi_B + \pi_W) = 1 - 2^{-1}(\pi'_B + \pi'_W),$$

we obtain

$$\begin{aligned}\varphi^{-1}H(v, y) &= v[v' - (y\pi'_B + y'\pi'_W)]\\ &\quad + v'[-v + 2^{-1}(\pi'_B + \pi'_W)]\\ &= v'2^{-1}(\pi'_B + \pi'_W) - v(y\pi'_B + y'\pi'_W).\end{aligned}$$

Thus, by Equation 12,

$$\varphi^{-1}\Delta v_n = v'_n2^{-1}(\pi'_B + \pi'_W) - E(V_n(Y_n\pi'_B + Y'_n\pi'_W)). \tag{21}$$

When $\pi_B = 1$, multiplication of Equation 20 by two and addition of Equation 21 yields

$$\begin{aligned}\varphi^{-1}\Delta v_n + 2\theta^{-1}\Delta y_n &= v'_n2^{-1}\pi'_W + E(V_nY'_n)\pi'_W\\ &= \pi'_WP(W_n).\end{aligned}$$

Theorem 3 follows when this equality is summed over $n \geq 0$.

In the remainder of the section, π_B is unrestricted. Since

$$Y'_n\pi'_W - Y_n\pi'_B = (\pi'_W + \pi'_B)(\ell - Y_n),$$

Equation 20 can be rewritten

$$\theta^{-1}\Delta y_n = (\pi'_W + \pi'_B)[\ell v_n - E(V_nY_n)].$$

Letting $n \longrightarrow \infty$, we obtain

$$E(V_\infty Y_\infty) = \ell v_\infty, \tag{22}$$

and, subtracting this from v_∞,

$$E(V_\infty Y'_\infty) = \ell' v_\infty. \tag{23}$$

Dividing Equation 21 by $\pi'_B + \pi'_W$ and letting $n \longrightarrow \infty$, we find that

$$0 = 2^{-1}v'_\infty - E(V_\infty Y_\infty)\ell' - E(V_\infty Y'_\infty)\ell.$$

In view of Equations 22 and 23,

$$0 = 2^{-1}v'_\infty - 2\ell\ell' v_\infty.$$

Hence

$$v_\infty = (1 + 4\ell\ell')^{-1},$$

and, using Equation 22 once again,

$$E(V_\infty Y_\infty) = \ell(1 + 4\ell\ell')^{-1}.$$

The proof of Theorem 2 is concluded by combining these equations with the limiting form of Equation 8.

8. SHIFTS

In recognition of the huge number of shift studies on children, I present the results concerning comparisons between shifts in this context. Consider, then, an experiment in the WGTA with stimuli varying along the dimensions of color, form, and position (see part 2 of Sec. 2). I assume that the subject is ignoring position, and, consequently, I have nothing further to say about this dimension. Suppose that, after reaching a strict criterion in Phase 1 of the experiment, *all subjects are shifted to a problem with form (triangle or circle) irrelevant, color (red or green) relevant, and red correct.* Assuming that there are only two learning rates, substituting green for W in Equation 19, and noting that $\pi_{\text{green}} = 0$, we obtain the basic formula

$$E(\#\text{green}) = \varphi^{-1}v'_0 + 2\theta^{-1}y'_0. \tag{24}$$

Here v is the probability of attending to color (rather than form), and y is the probability of choosing red, given attention to color:

$$v = P(\text{color}),\ y = P(\text{red} \mid \text{color}).$$

The quantities v_0 and y_0 are expected values of v and y on the initial trial of Phase 2.

The type of shift is defined by the stimulus and reinforcement conditions in Phase 1. For reversal (R), the same stimuli were used in Phase 1, color was relevant, but green was correct. As a first approximation, we shall assume that the child attends to color and chooses green with probability 1 at the beginning of Phase 2. Thus $v_0 = 1$ and $y_0 = 0$.

The intradimensional (I) shift also had color relevant in Phase 1, hence $v_0 = 1$. However there were different cues on both dimensions: e.g., blue and yellow, cross and square. Assuming no color generalization between phases, $y_0 = \frac{1}{2}$.

We want to compare both R and I to extradimensional (E) shift, in which form was relevant and color irrelevant in Phase 1. Thus $v_0 = 0$ and $y_0 = \frac{1}{2}$. In order to control for stimulus novelty in comparisons with R and I, it is desirable to retain Phase 1 cues in the first case and to introduce new cues in the second case. However the values of v_0 and y_0 are not sensitive to this variation in experimental procedure.

Values of v_0 and y_0 for the three shifts are summarized in Table 2, along with the corresponding expected number of errors from Equation 24.

TABLE 2
Values of v_0, y_0, and $E(\#\text{green})$ for three shifts

	v_0	y_0	$E(\#\text{green})$
R	1	0	$2\theta^{-1}$
E	0	1/2	$\varphi^{-1} + \theta^{-1}$
I	1	1/2	θ^{-1}

Denoting $E(\#\text{green})$ for R, E, and I by R*, E*, and I*, we note the following inequalities:

$$I^* < E^*, \tag{25}$$

$$E^* < R^* \quad \text{if } \varphi > \theta, \tag{26}$$

$$E^* > R^* \quad \text{if } \varphi < \theta. \tag{27}$$

As indicated in Section 1, one expects any attentional model to predict that I is easier than E. Thus the first inequality is not surprising. The second and third inequalities show that the relative difficulty of E and R is controlled by the relative rates of attentional and response learning. When response learning is faster, R is easier. Because R requires more response learning than E, this prediction is intuitively satisfying.

This comparison of I, E, and R would be of far greater interest if it did not presuppose only two learning rates. An important task for the future is to see how Equations 25, 26, and 27 must be modified in the general case.

The values of v_0 and y_0 in Table 2 are only approximations. To see this, note that they would not allow the effect of overtraining in Phase 1 to be taken into account. (An alternative viewpoint is that they presuppose substantial overtraining.) The treatment of the effect of overtraining on reversal in part 1 of the next section makes use of a more refined analysis of the values of v and y at the beginning of Phase 2.

9. THE OVERLEARNING REVERSAL EFFECT

The ZHL model will predict the ORE only when the values of its parameters ensure that perceptual learning profits more than response learning suffers as a result of overtraining. This condition can be met if the perceptual learning rates φ_i are smaller than the response-learning rates θ_j, or if these rates are of equal magnitude and V_0 is not too large. The influence of learning rates on overtraining is considered in part 1, the influence of V_0 in part 2. By taking account of both of these factors, Mackintosh (1969) was able to give a very comprehensive account of the experimental literature on the ORE in rats. This tour de force is discussed in part 3.

Throughout the section we consider a brightness discrimination in a jumping stand. Black is correct in Phase 1, white in Phase 2.

(1) *The influence of φ/θ.* Assume that there are only two learning rates, and suppose that

$$V_0 < 1 \quad \text{and} \quad Y_0 < 1 \tag{28}$$

a.s., where V_0 is the probability of attending to brightness on the initial trial of Phase 1, and Y_0 is the corresponding probability of choosing black, given attention to brightness. Let T_n be the expected total number of errors after a reversal on trial n, given V_n and Y_n. Applying Theorem 3 to reversal, we obtain

$$T_n = \varphi^{-1}V_n' + 2\theta^{-1}Y_n. \tag{29}$$

Note the appearance of Y_n rather than Y_n' on the right-hand side.

THEOREM 4. *If $\varphi < \theta$, T_n decreases for n sufficiently large. If $\varphi > \theta$, T_n increases for n sufficiently large.*

The statements of the theorem hold a.s. The first statement means that there is a (random) trial N beyond which each additional trial of training leads to fewer expected errors. This is a form of ORE. The magnitude of N surely depends on V_0 and Y_0 as well as φ and θ.

An earlier analytical study of the ORE dealt with $E(T_n)$ and was limited to small φ and θ. It found that $E(T_n)$ decreases over certain intervals $0 \ll N_1 < n < N_2$ for $\varphi/\theta < 3$, and increases for $\varphi/\theta > 3$ (Norman, 1972, Theorem 16.3.1). Though the 'critical value' of φ/θ is different from that in Theorem 4, the qualitative conclusion is the same: the ZHL model predicts the ORE for 'small' values of φ/θ but not for 'large' values.

Referring back to Equation 27, we see that the condition $\varphi < \theta$ under which the ORE is predicted is precisely the same as that under which reversal is easier than extradimensional shift. In the present experimental context, this means that an extradimensional shift from position to brightness should be more difficult than black-white reversal when and only when overtraining facilitates black-white reversal. One would like to know to what extent this prediction holds up when the unrealistic restriction of Equation 18 is dropped.

The proof of Theorem 4 is given at the end of the section.

(2) *The influence of* V_0. Lovejoy (1966) simulated the model in Table 1 with

$$\varphi_1 = \varphi_3 = \theta_1 = 0.07,$$

$$\varphi_2 = \varphi_4 = \theta_2 = 0.01.$$

These two values correspond to correct and incorrect responses, respectively. A variety of experimental findings, such as efficient performance on probabilistic schedules, require that the second value be smaller than the first. No stat-rats had initial response biases ($Y_0 = \frac{1}{2}$). For some, the relevant dimension was very obvious initially ($V_0 = 1$); for others it was not ($V_0 = 0.5$). Nonovertrained stat-rats were reversed after meeting a criterion of 15 consecutive correct responses. Overtrained animals had 150 additional trials between criterion and reversal. Sixty stat-rats were run under each combination of amount of training and initial obviousness. Table 3 gives the mean number of trials to criterion in reversal for each group.

TABLE 3
Trials to criterion in reversal in Lovejoy's simulation

	V_0	
	0.5	1.0
Nonovertrained	108.3	81.1
Overtrained	86.1	96.6

Differences between overtrained and nonovertrained animals were significant at the 2 percent level according to t tests within each column. The directions of these differences show that there is an ORE when $V_0 = 0.5$, but that overtraining hinders reversal when the relevant dimension is so obvious that it consistently attracts attention at the beginning of the experiment. Lovejoy (1966) showed that salience of the relevant dimension is also a crucial determinant of the effect of overtraining in experiments with real rats.

(3) *The joint influence of* φ_i, θ_j, *and* V_0. Mackintosh (1969) surveyed the experimental literature on overtraining and reversal in rats and concluded that, with few exceptions, the ORE occurred when and only when two conditions were met: the discrimination to be learned was fairly difficult and correct responses received large rewards. He also reported three new experiments whose results confirmed this generalization. The difficulty of a discrimination is a function of two variables: salience of the relevant and irrelevant dimensions (e.g., position is undoubtedly more obvious than brightness for naive rats) and distance between cues on the relevant dimension (e.g., a dark grey–light grey discrimination is more difficult than a black-white discrimination).

Mackintosh varied V_0 and θ_1 in the ZHL model to take account of problem difficulty and reward magnitude. As in Lovejoy's study, a larger value of V_0 corresponded to an easier problem. In addition, a larger value of θ_1 (which controls response learning after correct responses) was used to reflect a larger reward. Mackintosh simulated the model with the following parameter values:

$$Y_0 = 0.5;$$

$$V_0 = \begin{cases} 0.9, & \text{(easy problem)}, \\ 0.5, & \text{(hard problem)}; \end{cases}$$

$$\varphi_2 = \varphi_4 = \theta_2 = 0.01;$$

$$\varphi_1 = \varphi_3 = 0.10;$$

$$\theta_1 = \begin{cases} 0.10, & \text{(large reward)}, \\ 0.05, & \text{(small reward)}. \end{cases}$$

A group of 25 stat-rats was run under each combination of V_0 and θ_1 values. Each stat-rat was reversed twice, once with criterion values of V_n and Y_n and once with values obtained after 100 overtraining trials. The numbers of trials to criterion in reversal for each group, with and without overtraining, are presented in Mackintosh's Figure 2. Only the group that learned a hard problem with a large reward ($V_0 = 0.5$, $\theta_1 = 0.10$) showed an ORE.

I conclude that most experimental results relating the ORE to problem difficulty and reward magnitude can be interpreted within the framework of the ZHL model.

Proof of Theorem 4. We saw in part b of Section 4 that $\#\text{W} < \infty$ and $\#\text{po} < \infty$ a.s. Let L be the trial after the last occurrence of either W or po. (If $\#\text{W} = \#\text{po} = 0$, let $L = 0$.) Then $n \geq L$ implies $\text{br}_n\text{B}_n\text{C}_n$, so that, according to Table 1,

$$\begin{aligned} \Delta V_n &= \varphi V_n', \\ \Delta Y_n &= \theta Y_n'. \end{aligned} \tag{30}$$

It follows from Equation 30 that

$$V_n' = (1 - \varphi)^{n-L} V_L' = (1 - \varphi)^n V^*,$$

where

$$V^* = (1 - \varphi)^{-L} V_L'.$$

Substituting $(1 - \varphi)^n V^*$ for V_n' in Equation 30, we obtain

$$\Delta V_n = \varphi(1 - \varphi)^n V^*. \tag{31}$$

Since, by Equation 7, $\varphi < 1$, Equation 28 implies that $V_j' > 0$ for all $j \geq 0$. Hence $V_L' > 0$, and $V^* > 0$. Similarly,

$$\Delta Y_n = \theta(1 - \theta)^n Y^*, \tag{32}$$

where

$$Y^* = (1 - \theta)^{-L} Y_L' > 0.$$

Differencing Equation 29 and applying Equations 31 and 32, we obtain

$$\begin{aligned} \Delta T_n &= -\varphi^{-1}\Delta V_n + 2\theta^{-1}\Delta Y_n \\ &= -(1 - \varphi)^n V^* + 2(1 - \theta)^n Y^* \end{aligned}$$

for $n \geq L$. Thus $\Delta T_n < 0$ for n sufficiently large if $\varphi < \theta$, and $\Delta T_n > 0$ for n sufficiently large if $\varphi > \theta$.

10. APPROXIMATING $P(B_\infty)$

In this section I describe an approximation to $P(B_\infty)$ that is valid when the learning-rate parameters φ_i and θ_j are small so that learning is slow. If $\pi_B = 1$ and $\pi_W < 1$, as in part b of Section 4, then $P(B_\infty) = 1$; if $\pi_B < 1$ and $\pi_W = 1$, then $P(B_\infty) = 0$. Thus we can restrict our attention to the case where $\pi_B < 1$ and $\pi_W < 1$.

Letting $n \longrightarrow \infty$ in Equations 11, 12, and 8, we obtain

$$0 = E(V_\infty Q(Y_\infty)), \tag{33}$$

$$0 = E(H(V_\infty, Y_\infty)), \tag{34}$$

$$P(B_\infty) = E(V_\infty Y_\infty) + 2^{-1}v_\infty'. \tag{35}$$

If the asymptotic variances var (V_∞) and var (Y_∞) were zero, these equations would yield

$$0 = v_\infty Q(y_\infty),$$

$$0 = H(v_\infty, y_\infty),$$

$$P(B_\infty) = v_\infty y_\infty + 2^{-1}v_\infty'.$$

The first two equations could be solved for v_∞ and y_∞, and these could be substituted into the third to obtain $P(\mathrm{B}_\infty)$. In fact, var (V_∞) and var (Y_∞) are not zero, but they are small when φ_i and θ_j are small. Thus we expect this procedure to yield approximations to v_∞, y_∞, and $P(\mathrm{B}_\infty)$ that become more and more accurate as φ_i and θ_j approach 0.

Let us denote these approximations by v^*, y^*, and $P(\mathrm{B})^*$. The first two are defined to be the solutions of

$$v^* Q(y^*) = 0, \tag{36}$$

$$H(v^*, y^*) = 0, \tag{37}$$

such that $0 \leq v^*, y^* \leq 1$. The third is defined by

$$P(\mathrm{B})^* = v^* y^* + 2^{-1}(1 - v^*).$$

It can be shown that Equations 36 and 37 have unique solutions with $0 \leq v^*$, $y^* \leq 1$, and that, in fact, $0 < v^*, y^* < 1$. These solutions are quite easy to calculate. First, the quadratic equation

$$Q(y^*) = 0 \tag{38}$$

is solved for y^*; then the quadratic Equation 37 is solved for v^*.

Equations 37 and 38 are equivalent to $q(y^*) = 0$ and $h(v^*, y^*) = 0$, where q and h are obtained from Q and H by dividing by one of the learning-rate parameters, say θ_1. This has the effect of replacing all φ_is and θ_js in Equations 13 and 14 by the corresponding ratios φ_i/θ_1 and θ_j/θ_1. It follows that v^* and y^* depend only on ratios of learning-rate parameters, not on their absolute values. This suggests that we hold these ratios fixed as the learning-rate parameters approach zero.

THEOREM 5. *Suppose that the φ_is and θ_js approach* 0 *in such a way that their ratios remain constant. Then*

$$\operatorname{var}(V_\infty) \longrightarrow 0, \quad \operatorname{var}(Y_\infty) \longrightarrow 0,$$

$$v_\infty \longrightarrow v^*, \quad y_\infty \longrightarrow y^*,$$

and

$$P(\mathrm{B}_\infty) \longrightarrow P(\mathrm{B})^*. \tag{39}$$

Thus $P(\mathrm{B})^*$ will be a good approximation to $P(\mathrm{B}_\infty)$ when learning is slow. Table 4 illustrates the variation of $P(\mathrm{B})^*$ with reinforcement frequency and learning-rate parameters. The schedules are noncontingent ($\pi_\mathrm{B} = \pi'_\mathrm{W} = \pi$). As in Lovejoy's simulations (see part 2 of Sec. 9), it is assumed that

TABLE 4
$P(\mathrm{B})^*$ as a function of θ_1/θ_2 and π

θ_1/θ_2	π 0.50	0.60	0.70	0.80	0.90
1.0	0.50	0.55	0.61	0.68	0.79
2.5	0.50	0.59	0.68	0.79	0.90
5.0	0.50	0.64	0.77	0.88	0.95
10.0	0.50	0.73	0.87	0.93	0.97
20.0	0.50	0.83	0.93	0.97	0.99

$\varphi_1 = \varphi_3 = \theta_1$ and $\varphi_2 = \varphi_4 = \theta_2$. These triples of parameters correspond to reward and nonreward, respectively. Under these conditions, $P(\mathrm{B})^*$ depends only on π and θ_1/θ_2. The entries in the first column and first row of Table 4 are values of $P(\mathrm{B}_\infty)$ as well as $P(\mathrm{B})^*$. More generally, $P(\mathrm{B}_\infty) = P(\mathrm{B})^*$ if $\pi_\mathrm{B} = \pi_\mathrm{W}$ or if $\varphi_i = \varphi$ and $\theta_j = \theta$.

The most interesting feature of Table 4 is the steady increase of $P(\mathrm{B})^*$ for $\pi \geq 0.6$ as reward becomes more effective relative to nonreward. For $\theta_1/\theta_2 = 5$, $P(\mathrm{B})^*$ is slightly above π; for $\theta_1/\theta_2 = 20$ and $\pi \geq 0.7$, it is close to 1. When θ_1 and θ_2 are both small (and thus all φ_i are also small), $P(\mathrm{B}_\infty)$ should behave similarly in view of Equation 39. Therefore the ZHL model predicts efficient performance (i.e., large $P(\mathrm{B}_\infty)$) when learning is slow and reward is substantially more effective than nonreward.

It seems likely that small values of θ_2 will ensure fairly efficient performance even when θ_1 is not small.

Comments. Recall that K^∞ is the asymptotic joint distribution of V_n and Y_n as $n \longrightarrow \infty$ (see Sec. 4). As φ_i and θ_j approach zero in such a way that their ratios remain fixed, the distribution K^∞ approaches normality. A proof of this fact and of Theorem 5 can be constructed along the lines of the proof of Theorem 10.1.1(i) of Norman (1972). This represents an extension of the latter proof from one-process to two-process models.

Analogous results for the distribution of (V_n, Y_n) (n finite) are contained in Theorem 8.1.1 of Norman (1972). The approximations to v_n and y_n given by part (B) of that theorem are defined by a certain differential equation. The expected operator approximations considered by Zeaman and House (1963, pp. 173–175) are defined by analogous difference equations. A slight variant of the proof of Theorem 8.1.1(B) establishes the validity of expected operator approximations when φ_i and θ_j are small.

11. SOME RELATED MODELS

All-or-none learning. According to the first line of Table 1, the effect of attending to brightness, choosing black, and being correct on trial n is

$$\Delta V_n = \varphi_1 V_n' \quad \text{and} \quad \Delta Y_n = \theta_1 Y_n'.$$

Alternatively, we might assume that $V_{n+1} = 1$ if attentional conditioning is effective, $V_{n+1} = V_n$ if it is not effective, $Y_{n+1} = 1$ if response conditioning is effective, and $Y_{n+1} = Y_n$ if it is not effective. It remains to specify the probabilities of various combinations of effectiveness of conditioning in the two processes. The simplest possible assumptions are complete dependence and complete independence. In the first case, conditioning is either effective for both processes or ineffective for both processes, and these two events have probabilities c_1 and $1 - c_1$. In the second case, effective attentional conditioning has probability c_1, effective response conditioning has probability d_1, and these events occur simultaneously with probability $c_1 d_1$.

When comparable changes are made throughout Table 1, we obtain an all-or-none analog of the ZHL model. This all-or-none model might be of some interest in connection with experiments on human subjects. Though one expects it to be more mathematically tractable than the ZHL model, nothing is known about its predictions, except when effective conditioning occurs with probability 1. In that case, it reduces to the ZHL model with $\varphi_i = \theta_j = 1$.

Successive discrimination. This chapter has focused on a model for simultaneous discrimination, in which both cues on the relevant dimension (e.g., black and white) are present on each trial. In successive discrimination, only one cue is present on each trial. Thus, for a brightness discrimination in a jumping stand, the stimulus cards are either both white or both black. Right might be correct in the former case, left in the latter. Bush (1965) developed some ideas of L. B. Wyckoff, Jr., into a two-process model for successive discrimination that has much in common with the ZHL model. According to Bush's model, there is a probability v of observing the brightness cue, there are probabilities y_B and y_W of choosing right on black and white trials given the observing response, and there is a probability z of choosing right when the brightness cue is not observed. These probabilities undergo linear transformations on each trial. Some predictions of this model are described by Norman (1972, Sec. 17.2). The model's empirical adequacy has not been investigated.

An analogous all-or-none model was formulated somewhat earlier by Atkinson (1961). He applied it to an experiment in which college students could make overt observing responses as well as discrimination responses. Reinforcement was probabilistic. The model predicted asymptotic frequencies of both types of responses quite accurately.

12. CONCLUDING REMARKS

Though we have not systematically reviewed the experimental literature relevant to the ZHL model, we have seen that it is in qualitative agreement with data from a surprisingly wide variety of experiments. It has been claimed that multiple-cue learning in experiments with redundant relevant dimensions is inconsistent with 'one-look' models like the ZHL model; however, Shepp, Kemler, and Anderson (1972) have recently shown that this is not the case. This finding suggests that phenomena that appear damaging to these models must be interpreted with great care. Two such phenomena are errorless incidental learning (see Sutherland & Mackintosh, 1971, pp. 37–38; Shepp et al., 1972, pp. 326–327) and the tendency of rats to break position habits by choosing the correct visual cue (see Lovejoy, 1968, pp. 22, 45–46).

The ZHL model represents an extreme theoretical position, which doubtless has serious empirical limitations, but the nature of these limitations is not yet clearly understood. One reason for this is the lack of information concerning the quantitative accuracy of the model's predictions. No one has systematically estimated the model's parameters and compared its predictions with a variety of statistics from a suitable experiment. Only in this way can it be determined, for example, to what extent the magnitude and fine structure of position habits are compatible with the ZHL model. It would seem that the model has given a sufficiently accurate qualitative account of discrimination learning to merit the additional labor of quantitative testing.

REFERENCES

Atkinson, R. C. The observing response in discrimination learning. *Journal of Experimental Psychology*, 1961, **62**, 253–262.

Brunk, H. D. *An introduction to mathematical statistics*. (2nd ed.) Waltham, Mass.: Blaisdell, 1965.

Bush, R. R. Identification learning. In R. D. Luce, R. R. Bush, and E. Galanter (Eds.), *Handbook of mathematical psychology*. Vol. III. New York: Wiley, 1965.

Campione, J. C. The performance of preschool children on reversal and two types of extradimensional shifts. *Journal of Experimental Child Psychology*, 1971, **11**, 480–490.

Estes, W. K. *Learning theory and mental development*. New York: Academic Press, 1970.

Graf, V., & Tighe, T. Subproblem analysis of discrimination shift learning in the turtle (*Chrysemys picta picta*). *Psychonomic Science*, 1971, **25,** 257–259.

Kelleher, R. T. Discrimination learning as a function of reversal and nonreversal shifts. *Journal of Experimental Psychology*, 1956, **51,** 379–384.

Lovejoy, E. Analysis of the overlearning reversal effect. *Psychological Review*, 1966, **73,** 87–103.

Lovejoy, E. *Attention in discrimination learning*. San Francisco: Holden-Day, 1968.

Mackintosh, N. J. Extinction of a discrimination habit as a function of overtraining. *Journal of Comparative and Physiological Psychology*, 1963, **56,** 842–847.

Mackintosh, N. J. Further analysis of the overtraining reversal effect. *Journal of Comparative and Physiological Psychology Monograph*, 1969, **67,** 1–18.

Norman, M. F. Statistical inference with dependent observations: Extensions of classical procedures. *Journal of Mathematical Psychology*, 1971, **8,** 444–451.

Norman, M. F. *Markov processes and learning models*. New York: Academic Press, 1972.

Shepp, B. E., Kemler, D. G., & Anderson, D. R. Selective attention and the breadth of learning: An extension of the one-look model. *Psychological Review*, 1972, **79,** 317–328.

Sutherland, N. S., & Mackintosh, N. J. *Mechanisms of animal discrimination learning*. New York: Academic Press, 1971.

Zeaman, D., & House, B. J. The role of attention in retardate discrimination learning. In N. R. Ellis (Ed.), *Handbook of mental deficiency*. New York: McGraw-Hill, 1963.